

Broken S_3 Symmetry in Flavor Physics

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Abstract

The S_3 symmetry is shown to be a very good approximate symmetry when it is broken in a specific way. This is true both in quark sector and in lepton sector. The way to break it is implied by the K-M mechanism applied not to the mixing matrix but to the mass matrices. In quark sector, we have an almost perfect fitting to the experimental data, and in lepton sector, we have a precision for the θ_{13} .

Some time ago, one of the authors (H.S.) together with S. Pakvasa made a proposal to understand the flavor physics within the framework of S_3 symmetry[1]. Recently, we see some revival of this idea[2] together with possibilities of other discrete symmetry[3]. In this letter we show that S_3 group is a very good symmetry if it is broken in a specific way. This is true both in quark and in lepton sectors. We have a perfect fit to the CKM matrix in quark sector and some predictions in lepton sector: We have a one parameter description of the neutrino mixing matrix which becomes the tri-bimaximal[4] in the limit of vanishing θ_{13} . In fact, if we take the central value of the KamLAND data[6] on θ_{12} , we predict that the value of $\sin^2 \theta_{13}$ must be somewhere around 0.02.

It is well known by now that the three generation mass matrix (whether for quark or lepton sector) must have the following form if we assume the S_3 symmetry, the S_3 double-singlet quarks or leptons and the S_3 singlet Higgs:

$$M = \begin{pmatrix} a & b & b \\ b & a & b \\ b & b & a \end{pmatrix}. \quad (1)$$

This cannot be exact because, for example, it predicts at least two of the masses to be degenerate which is not true. The question is, therefore, what the proper way is to violate the S_3 symmetry. The clue is given by the K-M mechanism of CP violation. K-M mechanism[8] is to violate the CP by giving all possible phases to the quark mixing matrix. Since the mixing matrix is given by diagonalizing the mass matrices, we argue that the K-M mechanism should be applied to the more fundamental mass matrices rather than mixing matrix. This implies that we use, rather than (1), the following mass matrix:

$$M = \begin{pmatrix} ae^{i\delta_{11}} & be^{i\delta_{12}} & be^{i\delta_{13}} \\ be^{i\delta_{21}} & ae^{i\delta_{22}} & be^{i\delta_{23}} \\ be^{i\delta_{31}} & be^{i\delta_{32}} & ae^{i\delta_{33}} \end{pmatrix}. \quad (2)$$

Five of these phases can be absorbed in the wave functions leaving four phases and a, b as independent parameters. In the following we discuss the consequence of equation (2) both in the quark and the lepton sector.

(A) Quark sector

Starting from equation (2), we construct the MM^\dagger for which we get the following :

$$MM^\dagger = \begin{pmatrix} k & f & \bar{g} \\ \bar{f} & k & h \\ g & \bar{h} & k \end{pmatrix}. \quad (3)$$

with

$$k: \text{ real, } \text{Ref} \leq k, \text{ Reg} \leq k, \text{ Reh} \leq k \quad (4)$$

We can show that the form (3) with the condition (4) is equivalent to (2). In this case, only one phase can be absorbed in the wave function because the right handed quark wave function does not appear when we take the matrix element of (3). This leaves us 6 parameters as before. In fact we can absorb one more phase to the left handed up/down quark wave function but, by so doing we will not be able to absorb any phase from down/up quark wave function respectively, leaving always the 12 parameters. In the following we use the convention in which f and h of either up or down quark MM^\dagger are real.

The relation of the parameters to the quark masses is the following:

$$k = \frac{1}{3}s_1, \quad (5)$$

$$|f|^2 + |g|^2 + |h|^2 = \frac{1}{3}s_1^2 - s_2, \quad (6)$$

$$fgh + \bar{f}\bar{g}\bar{h} = \frac{2}{27}s_1^3 - \frac{1}{3}s_1s_2 + s_3 \quad (7)$$

with

$$s_1 = m_1^2 + m_2^2 + m_3^2, \quad s_2 = m_1^2m_2^2 + m_2^2m_3^2 + m_3^2m_1^2, \quad s_3 = m_1^2m_2^2m_3^2.$$

We use 6 quark masses[5] to reduce the number of parameters from 12 to 6. The remaining 6 parameters are fitted to quark mixing matrix (CKM matrix) which is written by 4 real parameters. The result of the fitting of the matrix MM^\dagger is given below. The convention we adopt is the reality of the parameters f and h of MM^\dagger of up quarks.¹

$$(MM^\dagger)_u = k_u \begin{pmatrix} 1.000 & 0.9999 & \left\{ \begin{array}{c} 0.9999 \\ -0.00001431i \end{array} \right\} \\ 0.9999 & 1.000 & 1.0000 \\ \left\{ \begin{array}{c} 0.9999 \\ +0.00001431i \end{array} \right\} & 1.0000 & 1.000 \end{pmatrix}$$

¹ More than 10 digits should be taken into account in order to reproduce quark mass spectrum, since their hierarchy structure is represented by small differences among values of order one.

$$(MM^\dagger)_d = k_d \begin{pmatrix} 1.000 & \begin{Bmatrix} 0.9955 \\ +0.08036i \end{Bmatrix} & \begin{Bmatrix} 0.9944 \\ +0.09255i \end{Bmatrix} \\ \begin{Bmatrix} 0.9955 \\ -0.08036i \end{Bmatrix} & 1.000 & \begin{Bmatrix} 0.9999 \\ +0.01278i \end{Bmatrix} \\ \begin{Bmatrix} 0.9944 \\ -0.09255i \end{Bmatrix} & \begin{Bmatrix} 0.9999 \\ -0.01278i \end{Bmatrix} & 1.000 \end{pmatrix}$$

These matrices give the following CKM matrix:

$$(\text{CKM}) = \begin{pmatrix} 0.9743 & 0.2253 & \begin{Bmatrix} 0.001249 \\ -0.003232i \end{Bmatrix} \\ \begin{Bmatrix} -0.2251 \\ -0.000136i \end{Bmatrix} & 0.9735 & 0.04102 \\ \begin{Bmatrix} 0.008026 \\ -0.003146i \end{Bmatrix} & \begin{Bmatrix} -0.04025 \\ -0.0007286i \end{Bmatrix} & 0.9992 \end{pmatrix}$$

Equivalently the Wolfenstein parameters[9] are given in table 1.

W-Parameters	Experimental	Calculated
λ	0.2253 ± 0.0007	0.225267
A	$0.808^{+0.022}_{-0.015}$	0.808417
$\bar{\rho}$	$0.132^{+0.022}_{-0.014}$	0.131926
$\bar{\eta}$	0.341 ± 0.013	0.340840

Table 1: Fitted Wolfenstein parameters.

Our convention for the CKM matrix is not the usual one but the perfection of our fitting is clear from the table of the convention-independent Wolfenstein parameters. The important point is that our mass matrices for up and down quarks (MM^\dagger matrices to be precise) are very close to the S_3 invariant ones indicating that the breaking of S_3 is small. In fact the imaginary part of f , h or g is always less than 10 % of the real part. It is also very impressive that the off-diagonal elements are all smaller than the diagonal element as the condition (4) indicates but the deviation is only less than 0.1 %. This corresponds to the smallness of $2 \otimes 2$ mass of $(2+1) \otimes (2+1)$ in S_3 implying the approximate democracy.

(B) Lepton sector

Lepton sector is more complicated than the quark sector due to the structure of the neutrino mass matrix. We adopt the generalized version of the see-saw mechanism[10] in the following way :

$$M_\nu^{(6)} = (\nu_L^T, (\nu_R^C)^T) C \begin{pmatrix} 0 & M^D \\ (M^D)^T & M^M \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^C \end{pmatrix}, \quad (8)$$

where $M^M = (M^M)^T$ is a 3×3 right handed Majorana mass matrix. The 6×6 eigenvalue equation becomes,

$$\begin{pmatrix} 0 & M^D \\ (M^D)^T & M^M \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \lambda \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} \quad (9)$$

For small λ (compared with the large eigenvalues of M^M which we assume), we get,

$$-M^D(M^M)^{-1}(M^D)^T V_1 = \lambda V_1 \quad \text{and} \quad V_2 \sim 0. \quad (10)$$

This implies the following extended see-saw equation,

$$M_\nu^{(3)} = -M^D(M^M)^{-1}(M^D)^T. \quad (11)$$

We assume that both M^D and M^M have the form of equation (2). Taking into account the fact that M^M is a complex symmetric matrix and also assuming that the S_3 breaking comes only from the phases of M^M , we write.

$$M^D = m_\nu \begin{pmatrix} 1 & c & c \\ c & 1 & c \\ c & c & 1 \end{pmatrix}, \quad (12)$$

$$M^M = a \begin{pmatrix} 1 & \eta e^{i\delta_1} & \eta e^{i\delta_2} \\ \eta e^{i\delta_1} & 1 & \eta e^{i\delta_3} \\ \eta e^{i\delta_2} & \eta e^{i\delta_3} & 1 \end{pmatrix}. \quad (13)$$

Substituting equations (12) and (13) into equation (11) we get,

$$M_\nu^{(3)} = \frac{m_\nu^2 \epsilon^2}{az} \left[\chi \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} - i \begin{pmatrix} -(x_1 + x_2) & x_1 & x_2 \\ x_1 & -(x_3 + x_1) & x_3 \\ x_2 & x_3 & -(x_2 + x_3) \end{pmatrix} \right], \quad (14)$$

where

$$\epsilon = c - 1, \quad \chi = 1 - \eta, \quad z = 1 + q_1 q_2 q_3 - q_1^2 - q_2^2 - q_3^2, \\ q_j = \eta e^{i\delta_j}, \quad x_j = \delta_{j+1} + \delta_{j+2} - \delta_j.$$

$M_\nu^{(3)}$ is generically a complex symmetric matrix and, according to the Takagi's factorization theorem[11], it can be diagonalized in the following way,

$$M_\nu^{(3)} = U_\nu^T \Sigma U_\nu,$$

where U_ν is a unitary matrix. In our particular case of equation (14), U_ν becomes an orthogonal matrix due to the fact that the real and the imaginary parts of $M_\nu^{(3)}$ commute with each other. We must distinguish two cases :

(1) Normal hierarchy case : $x_1x_2 + x_2x_3 + x_3x_1 = 0$

$$U_\nu = \left(\frac{1}{N_1} \begin{pmatrix} x_2 + 2x_3 \\ -(2x_2 + x_3) \\ x_2 - x_3 \end{pmatrix}, \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{N_3} \begin{pmatrix} x_2 \\ x_3 \\ -(x_2 + x_3) \end{pmatrix} \right) \quad (15)$$

The first column corresponds to an eigenvalue $-3m_\nu^2\epsilon^2\chi/(az)$, the second column to eigenvalue zero and the third one to $-m_\nu^2\epsilon^2\{3 - 2i(x_1 + x_2 + x_3)\}\chi/(az)$. We see that $x_2 = 0$ in (15) reduces exactly to tri-bimaximal case. Therefore, x_2 is the amount of the deviation of U_ν from the tri-bimaximal matrix.

(2) Inverted hierarchy: $x_1 = x_2 = x_3$

$$U_\nu = \begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix} \quad (16)$$

The first column corresponds to eigenvalue $-m_\nu^2\epsilon^2(3\chi - 3ix_2)/(az)$, the second column to $-m_\nu^2\epsilon^2\{3\chi - i(2x_1 + x_2)\}/(az)$ and the third column to zero eigenvalue. This does not correspond to the tri-bimaximal solution.

A few comments are in order regarding our results :

(1) There may be a correction from the phases of M^D which we assume to be vanishing, namely, the S_3 is violated only in the phases of M^M .

(2) The mixing matrix is actually $U_\nu \times U_l$ where U_l is a matrix diagonalizing the charged lepton mass matrix. The fact that U_ν alone gives the experimentally correct answer implies that U_l must be identity and M_l must be diagonal from the beginning. This means that the S_3 breaking in the charged lepton sector is not by the phase factor. This gives one important clue in constructing a model for flavor physics. One might think that there may be a solution in which neutrino mixing matrix is identity and the charged lepton mixing matrix gives the tri-bimaximal solution. We checked that this is not the case.

Next, we note that the mixing matrix given by equation (15) is a one parameter (x_2/x_3) description of the neutrino mixing matrix. The usual θ_{12} , θ_{23} and θ_{13} are all written by one parameter. We write this in the following

form in the case of small θ_{13} :

$$\sin^2 \theta_{13} = \kappa, \quad (17)$$

$$\tan^2 \theta_{12} = 0.5 + \frac{3}{4}\kappa, \quad (18)$$

$$\tan^2 \theta_{23} = 1 \pm 2\sqrt{2\kappa}. \quad (19)$$

Since we already have experimental values for θ_{12} and θ_{23} we should be able to predict the value of θ_{13} . Here we use the KamLAND data for the θ_{12} and equation (17) and (18) as an example to obtain θ_{13} . The fig. 1 shows that the value of $\sin^2 \theta_{13}$ is approximately 0.02. This is when we take the central value of KamLAND data[6] for $\tan^2 \theta_{12}$. We note that the central value of $\tan^2 \theta_{12}$ for the SuperK data[12] is less than 0.5 which is in contradiction to equation (18) combined with (17)².

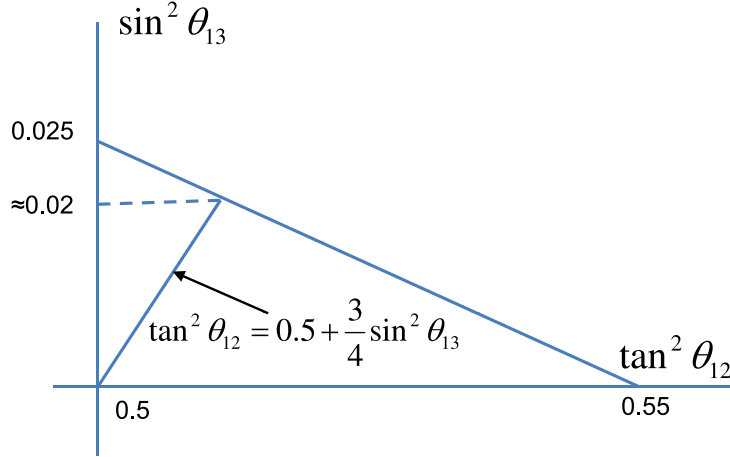


Figure 1: The value of $\sin^2 \theta_{13}$. KamLAND data is taken from Fig.5 of Ref. [7].

Finally, we make some brief comments on the model building based on our phenomenological analysis.

(1) The fact that we seem to have several independent phases imply that we may have more than one Higgs fields which provide appropriate phases.

² The most recent data of KamLAND[6] implies that $\tan^2 \theta_{12} > 0.5$ for the reactor neutrino but $\tan^2 \theta_{12} < 0.5$ when it is combined with the solar neutrino data.

$\sin^2 \theta_{12}$	$\tan^2 \theta_{12}$	$\sin^2 \theta_{23}$	$\tan^2 \theta_{23}$	$\sin^2 \theta_{13}$	$\tan^2 \theta_{13}$
0.3350	0.5038	0.5498	1.2213	0.0050	0.0050
0.3367	0.5076	0.5702	1.3266	0.0100	0.0101
0.3384	0.5115	0.5856	1.4134	0.0150	0.0152
0.3401	0.5155	0.5985	1.4908	0.0200	0.0204
0.3419	0.5195	0.6098	1.5625	0.0250	0.0256
0.3436	0.5236	0.6198	1.6302	0.0300	0.0309
0.3454	0.5277	0.6289	1.6949	0.0350	0.0363
0.3472	0.5319	0.6373	1.7574	0.0400	0.0417
0.3490	0.5362	0.6451	1.8181	0.0450	0.0471
0.3509	0.5405	0.6525	1.8774	0.0500	0.0526

Table 2: Mixing parameters for various values of $\kappa(= \sin^2 \theta_{13})$ from 0.005 to 0.05. (This corresponds to the positive sign in equation (19) which in turn implies $\tan \theta_{12} \sin \theta_{13} > 0$.)

We cannot, of course, deny that all the phases are connected to a single phase but it is not practical to construct such a model.

(2) The charged leptons belong to the 3 independent S_3 singlet whereas all the others including the right handed neutrinos belong to the singlet-doublet. The model must be able to explain this phenomenon.

(3) M^M and M^D are symmetric matrices. We checked that quark mass matrices are also complex symmetric matrices up to the accuracy of better than 0.6%. This suggests that the Higgs coupling to quarks or leptons must have the form $\bar{\Psi}_i \Psi_j H^{ij}$ where H^{ij} is symmetric in i and j ($i, j = 1, 2, 3$). When the vacuum value $\langle H^{ij} \rangle$ is such that it satisfies $\langle H^{11} \rangle = \langle H^{22} \rangle = \langle H^{33} \rangle$ and $\langle H^{ij} \rangle = v$ ($i \neq j$), we get the S_3 invariant mass matrix, if the vacuum values are all real. The S_3 breaking is provided by the phases of these vacuum values. A model can be easily and naturally constructed if we combine these phenomenological observations with the following theoretical argument: All the discrete symmetries and some global symmetries are an artifact of more fundamental gauge symmetry. Namely, they arise when some gauge symmetry is broken at some high energy. P, CP, R, Baryon number and lepton number all belong to this category[13] and so is S_3 . These global symmetries are valid below the energy of the breaking of original gauge symmetry. The simplest gauge symmetry we can think of in our case is SU_3 . We remind the reader that we get this group when we break E_8 to $E_6 \times SU_3$. This SU_3 can be shown to be infrared free[14] and its coupling may be very small at low energy but it could play a very important role in flavor physics.

We hope to report the result of such a model in a future publications.

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